Nova Série
BOLETIM
DA SOCIEDADE BRASILEIRA DE MATEMÁTICA

Bol. Soc. Bras. Mat., Vol.31, No. 3, 287-303

Maximal transitive sets with singularities for generic C^1 vector fields

C. M. Carballo, C. A. Morales and M. J. Pacifico*

Abstract. A transitive set Λ of a vector field X is *maximal transitive* if it contains every transitive set of X intersecting it. We shall prove that if X is C^1 generic then every singularity of X with either only one positive or only one negative eigenvalue belongs to a maximal transitive set of X. In particular, we characterize maximal transitive sets with singularities for generic C^1 vector fields on closed 3-manifolds in terms of homoclinic classes associated to a unique singularity. We apply our results to the examples introduced in [3] and [15].

Keywords: transitive sets, maximal transitive sets, transitive sets with singularities.

1 Introduction

Let M be a closed n-manifold and $\mathcal{X}^1(M)$ be the set of C^1 vector fields on M endowed with the C^1 topology. Given an open subset $\mathcal{A}$ of $\mathcal{X}^1(M)$ a subset $\mathcal{R} \subset \mathcal{A}$ is residual if it coincides with a countable intersection of open-dense subsets of $\mathcal{A}$. We say that *a generic vector field in $\mathcal{A}$ satisfies a property (P)* if there is a residual subset $\mathcal{R}$ of $\mathcal{A}$ such that (P) holds for every $X \in \mathcal{R}$. An invariant set of $X \in \mathcal{X}^1(M)$ is *transitive* if it is the ω-limit set of one of its points. A transitive set Λ of X is *maximal transitive* if it contains any transitive set T of X satisfying $T \cap \Lambda \neq \emptyset$.

Given a vector field X and a transitive set Λ of X, it is natural to ask about the existence of a maximal transitive set of X containing Λ. For example, every

Received 1 October 2000.
*This work is partially supported by CNPq 001/2000, FAPERJ and PRONEX/Dynamical Systems, FINEP-CNPq.

transitive set of an Axiom A vector field is contained in a maximal transitive set. There are examples of vector fields exhibiting transitive sets which are not contained in a maximal transitive set [11].

Assume that Λ is hyperbolic and isolated. Then we have two possibilities, namely Λ has singularities or not. If Λ has no singularities and X is C^1 generic, then Λ is contained in a maximal transitive set of X [4]. If Λ has singularities, then it reduces to a singularity. We say that a singularity is *co-dimension one* if it has either only one positive or only one negative eigenvalue. In this paper we shall prove the following result.

Theorem A. *A generic C^1 vector field X satisfies that every of its co-dimension one singularities is contained in a maximal transitive set of X.*

Corollary 1.1. *A generic three-dimensional C^1 vector field X satisfies that every of its singularities is contained in a maximal transitive set of X.*

The idea of the proof of Theorem A is the following. Denote by $H_X(p)$ the homoclinic class of a hyperbolic periodic orbit p of X. Recall that $H_X(p)$ is the closure of the transversal homoclinic points associated to p. It was proved in [4] that the homoclinic class associated to a periodic orbit p of a generic C^1 vector field X is maximal transitive. This was done as follows. First it was proved that, for every periodic orbit p of a generic C^1 vector field X, the closure $\mathrm{Cl}(W^u_X(p))$ of the unstable manifold $W^u_X(p)$ is a Lyapunov stable set of X. Similarly, for generic C^1 X, the closure $\mathrm{Cl}(W^s_X(p))$ of the stable manifold $W^s_X(p)$ of p is a Lyapunov stable set of the reversed flow $-X$. Next, it was proved that for every periodic orbit p of X generic, the following identity holds

$$H_X(p) = \mathrm{Cl}(W^u_X(p)) \cap \mathrm{Cl}(W^s_X(p)). \tag{1}$$

Finally it was noted that every transitive set of X realized as an intersection of a Lyapunov stable set of X with a Lyapunov stable set of $-X$ is a maximal transitive set.

This approach does not work to find maximal transitive sets in general, but (1) leads us to define, for any compact invariant set A of X,

$$H_X(A) = \mathrm{Cl}(W^u_X(A)) \cap \mathrm{Cl}(W^s_X(A)),$$

where

$$W^u_X(A) = \{q : \mathrm{dist}(X_t(q), A) \to 0, \quad t \to -\infty\}$$

and

$$W^s_X(A) = \{q : \mathrm{dist}(X_t(q), A) \to 0, \quad t \to \infty\}.$$

We believe that if A is a transitive set of a generic C^1 vector field X, then $H_X(A)$ defined as above is a maximal transitive set of X. Below we give sufficient conditions for $H_X(A)$ to be maximal transitive when A reduces to a singularity.

Theorem B. *If X is a generic C^1 vector field then the following conditions are equivalent:*

1. $W^u_X(\sigma) \subset H_X(\sigma)$.
2. $\mathrm{Cl}(W^u_X(\sigma)) = H_X(\sigma)$.
3. $\mathrm{Cl}(W^u_X(\sigma))$ *is transitive.*
4. $W^u_X(\sigma) \cap H_X(\sigma)$ *has non empty interior in* $W^u_X(\sigma)$.

Moreover, any of such a conditions implies that $H_X(\sigma)$ is a maximal transitive set. Similar result holds replacing u by s.

The above theorem implies the following characterization of maximal transitive sets with co-dimension one singularities for generic C^1 vector fields.

Corollary 1.2. *If X is a generic C^1 vector field, then Λ is a maximal transitive set with co-dimension one singularities of X if and only if $\Lambda = H_X(\sigma)$ for some co-dimension one singularity σ of X.*

Then, we have another corollary:

Corollary 1.3. *If X is a generic three-dimensional C^1 vector field, then Λ is a maximal transitive set with singularities of X if and only if $\Lambda = H_X(\sigma)$ for some singularity σ of X.*

The paper is organized as follows. Theorem A is proved using Theorem B. In Section 2 we shall prove both theorems. In Section 3 we give further applications of Theorem B. For the readers convinience we are including in the Appendix the results of [4] and [12] used here.

2 Proof of theorems A and B

In what follows M denotes a closed n-manifold, $n \geq 3$. Denote $\mathcal{X}^1(M)$ the space of C^1 vector fields endowed with the C^1 topology. We denote 2^M_c the set of compact subsets of M endowed with the Hausdorff topology. It follows that $\mathcal{X}^1(M)$ is a separable Banach space [5]. We shall say that $X \in \mathcal{X}^1(M)$ is *Kupka-Smale* if the periodic orbits and singularities of X are hyperbolic and the corresponding invariant manifolds intersect transversaly. The set of C^1 Kupka-Smale vector fields is denoted by $\mathcal{KS}^1(M)$. Recall that $-X$ denotes the time reversed vector field associated to $X \in \mathcal{X}^1(M)$.

We denote X_t the flow of $X \in \mathcal{X}^1(M)$, $t \in \mathbb{R}$. If $A \subset M$, $R \subset \mathbb{R}$ and $\epsilon > 0$ we denote $\mathrm{Cl}(A)$ the closure of A, $X_R(A) = \{X_r(a) : (r, a) \in R \times A\}$, and $B_\epsilon(A)$ the ϵ-ball centered at A. Set $\mathcal{O}_X(p) = X_{\mathbb{R}}(p)$ and $\mathcal{O}_X^+(p) = X_{[0,\infty)}(p)$ for any $p \in M$. The set of periodic orbits of X is denoted by $\mathrm{Per}(X)$, $\mathrm{Sing}(X)$ denotes the set of singularities of X, and $\mathrm{Crit}(X) = \mathrm{Per}(X) \cup \mathrm{Sing}(X)$.

If $\mathcal{Y}$ is a metric space, we say $\mathcal{R} \subset \mathcal{Y}$ is *residual* if $\mathcal{R}$ is a countable intersection of open-dense subsets of $\mathcal{Y}$. For example, $\mathcal{KS}^1(M)$ is residual in $\mathcal{X}^1(M)$. Clearly a countable intersection of residual subsets is residual.

A set-valued map

$$\Phi : \mathcal{Y} \to 2_c^M$$

is *lower semi-continuous* at $Y_0 \in \mathcal{R}$ if for every open set $U \subset M$ satisfying $U \cap \Phi(Y_0) \neq \emptyset$, there is a neighborhood $\mathcal{U}_0$ of Y_0 such that $U \cap \Phi(Y) \neq \emptyset$ for every $Y \in \mathcal{U}_0$. Similarly, Φ is *upper semi-continuous* at $Y_1 \in \mathcal{Y}$ if for every compact subset $K \subset M$ satisfying $K \cap \Phi(Y_1) = \emptyset$ there is a neighborhood $\mathcal{U}_1$ of Y_1 such that $K \cap \Phi(Y) = \emptyset$ for every $Y \in \mathcal{U}_1$. We say that Φ is lower semi-continuous if it does for every $Y_0 \in \mathcal{Y}$. A well known result [10] asserts that for suitable $\mathcal{Y}$, if Φ is lower semi-continuous then there is a residual subset $\mathcal{R}$ of $\mathcal{Y}$ such that Φ is also upper semi-continuous at every $Y_1 \in \mathcal{R}$.

A compact set $\Lambda \subset M$ is *Lyapunov stable* for X if for every neighborhood U of Λ there is a neighborhood $V \subset U$ of Λ such that $X_t(V) \subset U$ for every $t \geq 0$. The following criterion for Lyapunov stability is well known [2].

Lemma 2.1. *A compact set Λ is Lyapunov stable for X if it satisfies the following property*

(P) If $x_n \in M$ is a sequence converging to $x \in \Lambda$ and $t_n \geq 0$, then any limit point of the sequence $X_{t_n}(x_n)$ is in Λ.

Lemma 2.2. *Let Λ be a non empty compact set, $\Lambda = \Lambda^+ \cap \Lambda^-$, where Λ^+ is Lyapunov stable for X and Λ^- is Lyapunov stable for $-X$. If Λ^+ is transitive, then $\Lambda = \Lambda^+$. Similar property holds replacing $+$ by $-$.*

Proof. If Λ^+ is transitive and $\Lambda^+ \cap \Lambda^- \neq \emptyset$ with Λ^- Lyapunov stable for $-X$, using (P), we get that $\Lambda^+ \subset \Lambda^-$. So, $\Lambda^+ \cap \Lambda^- = \Lambda^+$ and we conclude $\Lambda = \Lambda^+$. □

The following lemma is a consequence of Lemma 2.1 and its proof is left for the reader.

Lemma 2.3. *Let Λ be non empty a compact set, $\Lambda = \Lambda^+ \cap \Lambda^-$, where Λ^+ is Lyapunov stable for X and Λ^- is Lyapunov stable for $-X$. If Λ is transitive then it is maximal transitive.*

Below we state a simple criterion for transitiveness using Lyapunov stability.

Lemma 2.4. *Let Λ be a transitive set of a vector field X. Suppose that there is $q \in \mathrm{Cl}(W^u_X(\Lambda)) \cap \mathrm{Cl}(W^s_X(\Lambda))$ such that $\omega_X(q)$ is Lyapunov stable for X. Then $\mathrm{Cl}(W^u_X(\Lambda)) = \omega_X(q)$. In particular, $\mathrm{Cl}(W^u_X(\Lambda))$ is a transitive set.*

Proof. Let Λ, q and X be as in the statement. Clearly, $\omega_X(q) \cap \mathrm{Cl}(W^s_X(\Lambda)) \neq \emptyset$. Let x be a point in this intersection. It can be approximated by a sequence x_n of points in $W^s(\Lambda)$. Then, we can choose a sequence t_n of positive times such that $\mathrm{dist}(X_{t_n}(x_n), \Lambda)$ goes to zero as $n \to \infty$. The sequence $y_n = X_{t_n}(x_n)$ has a subsequence that converges to a point y in Λ. As $\omega_X(q)$ is Lyapunov stable for X, we can apply Lemma 2.1 to show that y belongs to $\omega_X(q)$. Hence, $\Lambda \cap \omega_X(q) \neq \emptyset$. Using the transitivity of Λ we apply Lemma 2.1 again to get that $\Lambda \subseteq \omega_X(q)$. The Lyapunov stability of $\omega_X(q)$ also implies that the whole $W^u_X(\Lambda) \subseteq \omega_X(q)$. Then, since $\omega_X(q)$ is closed, the proof follows. □

Remark 2.5. *By [4] (see also the Appendix), under the hypothesis of Lemma 2.4, we obtain that for generic X, $\mathrm{Cl}(W^u_X(\Lambda))$ is a Lyapunov stable set of X.*

The following lemma is in [1], [7].

Lemma 2.6. *Let $X \in \mathcal{X}^1(M)$ and $x \in M \setminus (\mathrm{Per}(X) \cup \mathrm{Sing}(X))$. Suppose that for every $\delta > 0$ there are $x_p \in B_\delta(p)$, $x_q \in B_\delta(q)$, $t_p \geq 0$ and $t_q \leq 0$ such that $X_{t_p}(x_p) \in B_\delta(x)$ and $X_{t_q}(x_q) \in B_\delta(x)$. Then, for any C^1 neighborhood $\mathcal{U}$ of X in $\mathcal{X}^1(M)$, there is $L = L(\mathcal{U}) > 0$ so that for every $\epsilon > 0$ there exists $Y \in \mathcal{U}$ such that:*

1. *$Y = X$ outside $B_\epsilon(X_{[0,L]}(p)) \cup B_\epsilon(X_{[-L,L]}(x)) \cup B_\epsilon(X_{[-L,0]}(q))$ and*
2. *$q \in \mathcal{O}^+_Y(p)$.*

This lemma is used to prove the following proposition

Proposition 2.7. *There is a residual subset $\mathcal{R}$ of $\mathcal{X}^1(M)$ such that if $X \in \mathcal{R}$ and $\sigma \in \mathrm{Sing}(X)$, then the set*

$$\{p \in W^u_X(\sigma) : \mathrm{Cl}(\mathcal{O}^+_X(p)) \text{ is Lyapunov stable for } -X\}$$

is residual in $W^u_X(\sigma)$.

Let us introduce some useful notation before the proof of this proposition. Given $X \in \mathcal{KS}^1(M)$, then $\mathrm{Sing}(X)$ is a finite set and it is denoted by

$$\mathrm{Sing}(X) = \{\sigma_1(X), \cdots, \sigma_k(X)\}.$$

Denoting $\sigma_i(Y)$ the continuation of $\sigma_i(X)$ for Y close to X one has

$$\mathrm{Sing}(Y) = \{\sigma_1(Y), \cdots, \sigma_k(Y)\}. \tag{2}$$

for every Y close to X.

Let σ be a hyperbolic singularity of a vector field X. A *fundamental domain* of $W^u_X(\sigma)$ is a cross-section of $X/W^u_X(\sigma) - \{\sigma\}$ intersecting every orbit of $X/W^u_X(\sigma)$ (see [5]).

It is well known that the proof of Proposition 2.7 follows from the local result below [5].

Lemma 2.8. *For every $X \in \mathcal{KS}^1(M)$ there is a neighborhood $\mathcal{U}_X$ of X in $\mathcal{X}^1(M)$ and a residual subset $\mathcal{R}_X$ of $\mathcal{U}_X$ such that if $Y \in \mathcal{R}_X$, $\sigma \in \mathrm{Sing}(Y)$ and $D^u_Y(\sigma)$ is a fundamental domain of $W^u_Y(\sigma)$, then the set*

$$\{p \in D^u_Y(\sigma) : \mathrm{Cl}(\mathcal{O}^+_Y(p)) \text{ is Lyapunov stable for } Y\}$$

is residual in $D^u_Y(\sigma)$.

Proof. Let X be a Kupka-Smale C^1 vector field. As mentioned before it follows that there is a neighborhood $\mathcal{U}_X$ of X such that $\mathrm{Sing}(Y)$ satisfies (2) for every $Y \in \mathcal{U}_X$. For simplicity we denote $\dim(W^u_Y(\sigma_i(Y))) = u_i$ for $1 \le i \le k$.

Let $D^u_X(\sigma_i(X))$ be a fundamental domain of $W^u_X(\sigma_i(X))$. To simplify notation we shall assume that $D^u_X(\sigma_i(X))$ is the u_i-sphere S^{u_i}.

Let Σ_i be a cross-section of X such that $S^{u_i} = W^u_X(\sigma_i(X)) \cap \Sigma_i$. Shrinking $\mathcal{U}_X$, if necessary, we can assume that

$$D^u_Y(\sigma_i(Y)) = W^u_Y(\sigma_i(Y)) \cap \Sigma_i$$

is a fundamental domain of $W^u_Y(\sigma_i(Y))$ for every $Y \in \mathcal{U}_X$. It is suffices to prove the result for this particular fundamental domain.

By the Stable Manifold Theory [8] it follows that there is a C^1 map

$$\Gamma_i : \mathcal{U}_X \times S^{u_i} \to \Sigma_i$$

such that

$$D_Y^u(\sigma_i(Y)) = \text{Graph}(\Gamma_i(Y, \cdot)) = \{\Gamma_i(Y, y) : y \in S^{u_i}\}.$$

Note that the natural projection $\Pi_i : D_Y^u(\sigma_i(Y) \to S^{u_i}$, $\Pi_i(\Gamma_i(Y, y)) = y$, is a C^1 diffeomorphism.

We define the set valued map

$$\Phi_i : \mathcal{U}_X \times S^{u_i} \to 2_c^M$$

by

$$\Phi_i(Y, y) = \text{Cl}(\mathcal{O}_Y^+(\Gamma_i(Y, y))).$$

The Tubular Flow Box Theorem [5] and the continuity of Γ_i imply that Φ_i is lower semi-continuous. Denote $\mathcal{R}_i'$ the residual subset of $\mathcal{U}_X \times S^{u_i}$ such that Φ_i is upper semi-continuous in $\mathcal{R}_i'$.

Define, for every $Y \in \mathcal{U}_X$, the set

$$\mathcal{R}_i'(Y) = \{y \in S^{u_i} : (Y, y) \in \mathcal{R}_i'\}.$$

It follows that the set

$$\mathcal{V}_i = \{Y : \mathcal{R}_i'(Y) \text{ is residual in } S^{u_i}\}$$

is residual in $\mathcal{U}_X$.

Define

$$\mathcal{R}_X = \mathcal{KS}^1(M) \cap \left(\cap_{i=1}^k \mathcal{V}_i\right).$$

Clearly $\mathcal{R}_X$ is a residual subset of $\mathcal{U}_X$. If $Y \in \mathcal{R}_X$ we define

$$G_i(Y) = \{\Gamma_i(Y, y) : y \in \mathcal{R}_i'(Y)\}.$$

It follows that $G_i(Y)$ is residual in $D_Y^u(\sigma_i(Y))$, $\forall i = 1, \cdots, k$.

The proof of the proposition is then reduced to prove that $\text{Cl}(\mathcal{O}_Y^+(p))$ is Lyapunov stable for Y, $\forall (i, Y, p) \in \{1, \cdots, k\} \times \mathcal{R}_X \times G_i(Y)$. For this we assume by contradiction that there is $(i, Y, p) \in \{i, \cdots, k\} \times \mathcal{R}_X \times G_i(Y)$ such that $\text{Cl}(\mathcal{O}_Y^+(p))$ is not Lyapunov stable for Y. Note that there is $y \in \mathcal{R}_i'(Y)$ such that $\Phi_i(Y, y) = \text{Cl}(\mathcal{O}_Y^+(p))$. In particular $(Y, y) \in \mathcal{R}_i'$, i.e. Φ_i is upper semi-continuous at (Y, y).

By Lemma 2.1 there are sequences $x_n \to x \in \mathrm{Cl}(\mathcal{O}_Y^+(p))$ and $t_n > 0$ such that $X_{t_n}(x_n) \to q \notin \mathrm{Cl}(\mathcal{O}_Y^+(p))$. Observe that $p \notin \mathrm{Crit}(X)$. We can assume that $x \notin \mathrm{Crit}(X)$ for, otherwise, we could replace x by some point in $W_Y^u(x) \setminus \{x\}$ (recall that Y is Kupka-Smale). Similarly we can assume that $q \notin \mathrm{Crit}(Y)$ for, otherwise, we could replace q by some point in $W_Y^s(q) \setminus \{q\}$.

Let U be a neighborhood of $\mathrm{Cl}(\mathcal{O}_Y^+(p))$ satisfying $q \notin U$. In particular, $K = M \setminus U$ is compact, $\Phi_i(Y, y) \cap K = \emptyset$ and $q \in K$. As $x \in \mathrm{Cl}(\mathcal{O}_Y^+(p))$, there is a sequence $s_n \geq 0$ such that $X_{s_n}(p) \to x$. Hence, for every $\delta > 0$ there are $x_p = p \in B_\delta(p)$, $x_q = X_{t_n}(x_n) \in B_\delta(x)$, $t_p = s_n \geq 0$ and $t_q = -t_n \leq 0$ such that $X_{t_p}(x_p) \in B_\delta(x)$ and $X_{t_q}(x_q) \in B_\delta(x)$. Then, by Lemma 2.6 there is $Z \in \mathcal{X}^1(M)$ arbitrarily C^1 close to Y such that $q \in \mathrm{Cl}(\mathcal{O}_Z^+(p))$. This last fact contradicts the upper semi-continuity of Φ_i at (Y, y) for $q \in \Phi_i(Z, p) \cap K \neq \emptyset$. This contradiction concludes the proof. □

The following lemma is not difficult, it is in [12], and it is proved in the Appendix for completeness.

Lemma 2.9. *If $X \in \mathcal{X}^1(M)$ and $p \in M$, then $\mathrm{Cl}(\mathcal{O}_X^+(p))$ is Lyapunov stable for X if and only if $\omega_X(p)$ does.*

Proof of Theorem B. Let X be a generic C^1 vector field and $\sigma \in \mathrm{Sing}(X)$. By [4] we can assume that $\mathrm{Cl}(W_X^u(\sigma))$ is Lyapunov stable for X and $\mathrm{Cl}(W_X^s(\sigma))$ is Lyapunov stable for $-X$. Recall that $H_X(\sigma) = \mathrm{Cl}(W_X^u(\sigma)) \cap \mathrm{Cl}(W_X^s(\sigma))$ by definition.

Clearly (1) implies (2) since $H_X(\sigma)$ is both closed and contained in $\mathrm{Cl}(W_X^u(\sigma))$.

Now assume that (2) holds. By Proposition 2.7 there is $p \in \mathrm{Cl}(W_X^u(\sigma)) = H_X(\sigma) = \mathrm{Cl}(W_X^u(\sigma)) \cap \mathrm{Cl}(W_X^s(\sigma))$ such that $\mathrm{Cl}(\mathcal{O}_X^+(p))$ is Lyapunov stable for X. Then, by Lemma 2.4 and Lemma 2.9 applied to $\Lambda = \{\sigma\}$, we obtain that $\mathrm{Cl}(W_X^u(\sigma))$ is transitive, proving (3).

If (3) holds it follows that $H_X(\sigma) = \mathrm{Cl}(W_X^u(\sigma))$ by Lemma 2.2. Thus, $W_X^u(\sigma) \cap H_X(\sigma) = W_X^u(\sigma)$, proving (4).

If (4) holds there is $p \in H_X(\sigma)$ such that $\mathrm{Cl}(\mathcal{O}_X^+(p))$ is Lyapunov stable for X by Proposition 2.7. Then $\mathrm{Cl}(W_X^u(\sigma))$ is transitive by Lemma 2.4 and Lemma 2.9. By Lemma 2.2 we conclude that $H_X(\sigma) = \mathrm{Cl}(W_X^u(\sigma))$, proving (1).

Clearly, $H_X(\sigma)$ is transitive if one of the above conditions hold. In particular, any of the conditions (1)-(4) implies that $H_X(\sigma)$ is maximal transitive. Indeed, as $\mathrm{Cl}(W_X^u(\sigma))$ is Lyapunov stable for X and $\mathrm{Cl}(W_X^s(\sigma))$ is Lyapunov stable for $-X$ Lemma 2.1 implies that $H_X(\sigma)$ contains any transitive set T of X satisfying $H_X(\sigma) \cap T \neq \emptyset$.

A similar argument proves an analogous result replacing u by s. This completes the proof. □

Proof of Theorem A. Let X C^1 generic and $\sigma \in \mathrm{Sing}(X)$ be of co-dimension one. We assume $\dim W^u_X(\sigma) = 1$, otherwise we consider $-X$ instead of X. By [4] (see also the Appendix) we can further assume that $\mathrm{Cl}(W^s_X(\sigma))$ is Lyapunov stable for $-X$ and $\mathrm{Cl}(W^u_X(\sigma))$ is Lyapunov stable for X. Since $H_X(\sigma) = \mathrm{Cl}(W^s_X(\sigma)) \cap \mathrm{Cl}(W^s_X(\sigma))$, if $H_X(\sigma) = \{\sigma\}$ we obtain, by Lemma 2.3, that $H_X(\sigma)$ is maximal transitive. So, we assume $H_X(\sigma) \setminus \{\sigma\} \neq \emptyset$. In this case we shall prove that $W^u_X(\sigma) \subset H_X(\sigma)$. Indeed, let $x \in H_X(\sigma) \setminus \{\sigma\}$. In particular, $x \in \mathrm{Cl}(W^u_X(\sigma)) \setminus \{\sigma\}$. As $\dim(W^u_X(\sigma)) = 1$ it follows that $x \in \omega_X(p)$, for some $p \in W^u_X(\sigma)$. Since $\dim W^u_X(\sigma) = 1$ we have, by Proposition 2.7 and Lemma 2.9, that $\omega_X(p)$ is Lyapunov stable for X. On the other hand, as $\mathrm{Cl}(W^s_X(\sigma))$ is Lyapunov stable for $-X$ and $\omega_X(p) \cap \mathrm{Cl}(W^s_X(\sigma)) \neq \emptyset$ (because $x \in \omega_X(p) \cap \mathrm{Cl}(W^s_X(\sigma))$), by Lemma 2.1, we get $p \in \mathrm{Cl}(W^s_X(\sigma))$. Hence $p \in W^u_X(\sigma) \cap \mathrm{Cl}(W^s_X(\sigma)) \subset H_X(\sigma)$, implying that $\omega_X(p) \subset H_X(\sigma)$. Since $\omega_X(p)$ is Lyapunov stable for X and $\omega_X(p) \cap \mathrm{Cl}(W^s_X(\sigma)) \neq \emptyset$ we obtain, by Lemma 2.1, that $\sigma \in \omega_X(p)$. Using again that $\omega_X(p)$ is Lyapunov stable for X we obtain $W^u_X(\sigma) \subset \omega_X(p) \subset H_X(\sigma)$. By Theorem B we conclude the proof of Theorem A.

3 Applications

In this section we shall discuss an application of Theorem B concerning the persistence of attractors in the C^1 topology. The definition of attractor we deal with is the following. A compact invariant set Λ of a vector fields X is an *attracting set* if there is an open set U (called isolating block) such that $X_t(U) \subset U$ for every $t > 0$ and

$$\Lambda = \cap_{t\in\mathbb{R}} X_t(U).$$

An *attractor* is a transitive attracting set (this differs from [9] where transitive attracting sets were called Thom attractors). A repeller of X is an attractor for $-X$. An attractor (a repeller) which reduces to a periodic orbit or singularity is called *sink* (*source*).

It would be interesting to characterize attractors which are "robust" under small perturbations. There are several definitions for robustness of attractors among which we can mention the following one. An attractor Λ of a C^r vector field X is C^r *robust*, $r \geq 1$, if there is an isolating block U of Λ such that, for every Y C^r close to X, $\cap_{t\in\mathbb{R}} Y_t(U)$ is a nontrivial attractor of Y.

The above definition of robust attractor is related with the notion of partial hyperbolicity. Recall that a compact invariant set Λ of a C^1 vector field is *partially hyperbolic* if it exhibits a nontrivial continuous splitting $E^s \oplus E^c$, invariant by the derivative DX_t, such that E^s is contracting by DX_t and dominates E^c, that is, there are constants $0 < \lambda < 1$, $\gamma > 0$ such that for any $t > 0$, $\|DX_t/E^s\| \, \|DX_{-t}/E^c\| < \gamma\lambda^t$. We say that the central direction E^c is volume expanding if DX_t restricted to E^c is volume expanding.

In [13] it was proved that partial hyperbolicity with volume expanding central direction is a necessary condition for an attractor of a three-dimensional C^1 vector field to be C^1 robust. However, partially hyperbolicity is not a sufficient condition for robustness of attractors as an example in [14] shows. So, it would be interesting to find weaker notions of robustness leading to a classification in terms of partial hyperbolicity. In this section we deal with the following one. An attracting set Λ of a C^r vector field X is *C^r-weakly robust* if there are an isolating block U of Λ and a C^r-neighborhood $\mathcal{U}$ of X such that the following set

$$\{Y \in \mathcal{U} : Y \text{ has no sources in } U \text{ and there is a nontrivial maximal transitive set } T \subset U \text{ of } Y \text{ such that } W^s_Y(T) \cap U \text{ is residual in } U\}$$

is residual in $\mathcal{U}$. An attractor is C^r-weakly robust if it is C^r-weakly robust as attracting set.

It is clear that C^r robust attractors are C^r-weakly robust ones. The example in [14] mentioned before is a non-robust attractor which is weakly robust.

The following result gives a sufficient condition for an attracting set to be C^1-weakly robust.

Proposition 3.1. *The condition (H) below suffices for an attracting set Λ of a C^1 vector field X to be C^1-weakly robust.*

(H) There are an isolating block U of Λ and a C^1 neighborhood $\mathcal{U}$ of X such that, for every $Y \in \mathcal{U}$, Y has no sources in U and there is $p \in \text{Crit}(Y) \cap U$ such that $W^s_Y(p) \cap U$ is dense in U.

Proof. Let Λ be an attracting set of a C^1 vector field X and suppose that Λ satisfies (H). Let $H_X(p)$ be the homoclinic class of p if p is a periodic orbit of X or $H_X(p) = \text{Cl}(W^u_Y(p)) \cap \text{Cl}(W^s_Y(p))$ if $p \in \text{Sing}_Y(\Lambda)$. Recall that the homoclinic class of $p \in \text{Per}(X)$ is the closure of the set of transverse homoclinic orbits associated to p. By [4] we have $H_Y(p) = \mathrm{Cl}(W^u_Y(p)) \cap \mathrm{Cl}(W^s_Y(p))$ for generic $Y \in \mathcal{U}$. We claim that for generic $Y \in \mathcal{U}$,

$$H_Y(p) = \mathrm{Cl}(W^u_Y(p)). \tag{3}$$

Indeed, $\mathrm{Cl}(W^u_Y(p)) \subset U$ for every $Y \in \mathcal{U}$ since U is an isolating block of Λ. Thus, $\mathrm{Cl}(W^s_Y(p)) \supset \mathrm{Cl}(W^u_Y(p))$ by (H). So, for generic $Y \in \mathcal{U}$,

$$\mathrm{Cl}(W^u_Y(p)) \supset H_Y(p) = \mathrm{Cl}(W^u_Y(p)) \cap \mathrm{Cl}(W^s_Y(p)) \supset \mathrm{Cl}(W^u_Y(p))$$

proving the claim. Then, by Theorem B, $H_Y(p)$ is transitive. Thus, by [4], there is $\mathcal{R} \subset \mathcal{U}$ residual such that $T = \mathrm{Cl}(W^u_Y(p))$ is a transitive Lyapunov stable set of Y, $\forall Y \in \mathcal{R}$.

It remains to prove that $W^s_Y(T) \cap U$ is residual in U, for every $Y \in \mathcal{R}$. For this we proceed as follows. As $p \in T$, (H) implies that $W^s_Y(T) \cap U$ is dense in U. Since T is Lyapunov stable, there is a sequence $U_n \subset \mathrm{Cl}(U_n) \subset U_{n-1}$, $n \geq 1$, of positively invariant open sets such that $\cap_{n\geq 1} U_n = T$. We claim that $W^s_Y(U_n)) \cap U$ is open and dense in U. Indeed, let $B_{n-1} = \{q \in W^s_Y(U_{n-1}) \cap U; \omega_Y(q) \subset \mathrm{Cl}(U_n)\}$. B_{n-1} is open and dense in U. Clearly $W^s_Y(U_n) \cap U \supset B_n$. Hence $W^s_Y(T) = \cap_n W^s_Y(U_n) \cap U \supset \cap_n B_n$ is residual in U. Thus, $W^s_Y(T) \cap U$ is residual in U. The proof is completed. □

Let us describe some examples of attracting sets where Proposition 3.1 applies.

1. Attractors with several expanding directions ([3]). Let $f^* : T^k \to T^k$ an expanding map on T^k, the k-dimensional torus. That is f^* is a C^r map in T^k for which there is a constant $\lambda > 1$ such that $\|Df^*(x)\| \geq \lambda$. Denote D^2 the two-dimensional disk. It follows that $T^k \times D^2$ can be realized as an isolating block of an hyperbolic attractor A in a way that the vertical foliation $V = \{\{q\} \times D^2 : q \in T^k\}$ is contracting. Suspending A we obtain a $(k+3)$-dimensional C^r vector field X' with a hyperbolic attractor Λ'. Note that the open set $U' = T^k \times D^2 \times S^1$ is an isolating block of Λ' with cross-section $\Sigma = T^k \times D^2$. Following [3] one can modify Λ' around a tubular neighborhood in order to obtain a C^r vector field X and an attracting set Λ of X having a hyperbolic singularity σ with several expanding directions. Note that σ is not codimension one any longer. In addition, Λ has Σ as a cross section with expanding return map $F : \Sigma_0 \to \Sigma$, where $\Sigma_0 = \Sigma \setminus C$ for some subset C of $W^s_X(\sigma)$. Note that the vertical foliation V is invariant and contracting for F. The modification can be done in a way that there is a quotient map $f : \Sigma_0/V \to \Sigma/V$ satisfying $\|Df(x)\| \geq \lambda$ for every $x \in \Sigma_0/V$. It was proved in [3] that if Δ is the diameter of T^k, R is the radius of injectivity of the exponential map in T^k and the constant λ satisfies

$$\lambda > 2\max\left\{1, \frac{\Delta}{R}\right\}, \tag{4}$$

then the quotient map f is transitive. Consequently, Λ is a C^r robust attractor of X if the above inequality holds.

It is natural to consider the case $\lambda \in (1,2)$ in the above setting (see Remark 3.3 below). Applying Proposition 3.1 one can prove the following fact.

Proposition 3.2. *If $\lambda > 1$, then Λ is a C^1-weakly robust attracting set of X.*

Proof. As $\lambda > 1$ it is immediate that Λ satisfies (H). Then, the result follows from Proposition 3.1. □

Remark 3.3. *Proposition 3.2 can be seem as an indication that the following result is true: Let f be a C^1 expanding map defined on $M \setminus \{c\}$ for some $c \in M$, where M is a n compact manifold. Then, there is a closed and connected set $J \subset M$ containing c in its interior such that for every $x \in J \setminus \{c\}$ there is a return time $n \in \mathbb{N}$ such that the induced return map $x \mapsto f^n(x)$ is transitive. By an expanding map we mean that there is $\lambda > 1$ such that $\|Df(x)\| > \lambda$ for every $x \in M$. See Theorem 2.1 in [14] for a one-dimensional version of this result.*

2. *Wild strange attractors [15].* In [15] it was constructed a four-dimensional vector field X exhibiting an open set D and a singularity $O \in D$ such that

(a) X is transverse to the boundary of D and $\mathrm{Cl}(X_t(D)) \subset D$ for every $t > 0$.

(b) X has a cross-section $\Pi \subset D$ intersecting every nonsingular flowline of X in D.

(c) The eigenvalues $\gamma, -\lambda \pm i\omega, -\alpha$ of $DX(O)$ satisfy $\gamma, \lambda, \omega, \alpha \in \mathbb{R}$, $\gamma > 0$, $0 < \lambda < \alpha$, $\omega \neq 0$.

(d) X has a dominated splitting $E^s \oplus E^c$ in D such that E^s is one-dimensional. Moreover, X_t contracts E^s and expands volume along E^c for $t > 0$.

Althought [15] assumed that X is C^r, $r \geq 4$, such a construction can be also done in the C^1 topology.

It is clear that the above conditions (a)-(d) are open, i.e. they are satisfied for every Y in a C^1 neighborhood $\mathcal{V}$ of X (obviously replacing X by Y and O by $O(Y)$).

For every $Y \in \mathcal{V}$ we define

$$\Lambda(Y) = \cap_{t>0} Y_t(D).$$

Clearly $\Lambda(Y)$ is an attracting set of Y. Let $A(Y)$ be the set of $q \in D$ accessible from $O(Y)$. Recall that q is accessible from $O(Y)$ if for every $\epsilon > 0$ there is a $(\epsilon, 1)$-orbit of Y joining $O(Y)$ to q (see [15] for details).

Note that (a) implies that $A(Y) \subset \Lambda(Y)$ for every $Y \in \mathcal{V}$. By Lemma 2 in [15] condition (H) is satisfied with $U = D$, $\mathcal{U} = \mathcal{V}$, $\Lambda = \Lambda(Y)$, and $p = O(Y)$. And using (H) it was proved in [15] that $A(Y)$ is a *chain transitive* Lyapunov stable set such that $W_Y^s(A(Y)) \cap D$ is residual in D for every $Y \in \mathcal{V}$. On the other hand, using directly Proposition 3.1 we conclude that $\Lambda(Y)$ is a C^1-weakly attracting set for every $Y \in \mathcal{V}$. Thus generic $Y \in \mathcal{V}$ exhibits a *transitive* Lyapunov stable set $T(Y) \subset A(Y)$ such that $W^s(T(Y)) \cap D$ is residual in D.

4 Appendix

In this section we include results of [4] and [12] used in the paper.

The next proposition is in [4] and its proof uses the same tools as Lemma 2.8.

Proposition 4.1. *There exists a residual set $\mathcal{R}$ of $\mathcal{X}^1(M)$ such that, for every $Y \in \mathcal{R}$ and $\sigma \in$ Crit(Y), Cl$(W_X^u(\sigma))$ is Lyapunov stable for X and Cl$(W_X^s(\sigma))$ is Lyapunov stable for $-X$.*

Proof. Let us start with some notations. Given $X \in \mathcal{X}^1(M)$ and $p \in \text{Per}(X)$ we denote $\Pi_X(p)$ the period of p. It is convenient to consider a singularity as a periodic orbit with period zero.

If $T > 0$ we denote

$$\text{Crit}_T(X) = \{p \in \text{Crit}(X) : \Pi_X(p) < T\}.$$

If $p \in \text{Crit}(X)$ is hyperbolic, then there is a continuation $p(Y)$ of p for Y close enough to X so that $p(X) = p$.

Note that if $X \in \mathcal{KS}^1(M)$ and $T > 0$, then

$$\text{Crit}_T(X) = \{p_1(X), \cdots, p_k(X)\}$$

is a finite set. Moreover,

$$\mathrm{Crit}_T(Y) = \{p_1(Y), \cdots, p_k(Y)\}$$

for every Y close enough to X.

Clearly Proposition 4.1 is a consequence of the following lemma [5].

Lemma 4.2. *If $X \in \mathcal{KS}^1(M)$ and $T > 0$ then there is a neighborhood $\mathcal{U}_{X,T}$ of X, and a residual subset $\mathcal{R}_{X,T}$ of $\mathcal{U}_{X,T}$, so that if $Y \in \mathcal{R}_{X,T}$ and $p \in \mathrm{Crit}_T(Y)$ then $\mathrm{Cl}(W^u_Y(p))$ is Lyapunov stable for Y and $\mathrm{Cl}(W^s_Y(p))$ is Lyapunov stable for $-Y$.*

Proof. As already mentioned, $\mathrm{Crit}(Y) = \{p_1(Y), \cdots, p_k(Y)\}$ for every Y in some neighborhood $\mathcal{U}_{X,T}$ of X.

For any $i \in \{1, \cdots, k\}$ we define $\Phi_i : \mathcal{U}_{X,T} \to 2^M_c$ by

$$\Phi_i(Y) = \mathrm{Cl}(W^u_Y(p_i(Y)).$$

By the continuous dependence of unstable manifolds we have that Φ_i is a lower semi-continuous map, and so, Φ_i is also upper semi-continuous for every vector field in some residual subset $\mathcal{R}_i$ of $\mathcal{U}_{X,T}$. Set $\mathcal{R}_{X,T} = \mathcal{KS}^1(M) \cap (\cap_i \mathcal{R}_i)$. Then $\mathcal{R}_{X,T}$ is residual in $\mathcal{U}_{X,T}$. Let us prove that $\mathcal{R}_{X,T}$ satisfies the conclusion of the lemma.

Let $\sigma \in \mathrm{Crit}_T(Y)$ for some $Y \in \mathcal{R}_{X,T}$. Then, $\sigma = p_i(Y)$ for some i, and so, $\Phi_i(Y) = \mathrm{Cl}(W^u_Y(\sigma))$.

Suppose by contradiction that $\mathrm{Cl}(W^u_Y(\sigma))$ is not Lyapunov stable for Y. Then, there are sequences $x_n \to x \in \mathrm{Cl}(W^u_Y(\sigma))$ and $t_n \geq 0$ such that

$$q = \lim_{n\to\infty} Y_{t_n}(x_n) \notin \mathrm{Cl}(W^u_Y(\sigma)). \tag{5}$$

We have either

(a) $x \notin \mathrm{Crit}(Y)$ or

(b) $x \in \mathrm{Crit}(Y)$.

It is enough to prove case (a). Indeed, if x is as in (b), it can be neither an attracting nor a repelling singularity or periodic orbit, and so $W^u_Y(x) \setminus \{x\} \neq \emptyset$. As $x_n \to x$, there is $r \in W^u_Y(x)$ such that $x_n \to r$. We claim that $r \in \mathrm{Cl}(W^u_Y(\sigma))$. Otherwise, using the Connecting Lemma [6] we obtain Z C^1 near to Y such that $W^u_Z(\sigma(Z)) \cap W^s_Z(x) \neq \emptyset$, producing a saddle-connection for Z. By another C^1 perturbation Z' of Z we break this saddle-connection, to send $W^u_{Z'}(\sigma(Z'))$

close to r contradicting the fact that Φ_i is upper semi-continuous at Y. Thus, $r \in \mathrm{Cl}(W_Y^u(\sigma))$. Since $r \notin \mathrm{Crit}(Y)$, we conclude as in case (a), replacing x by r.

Now we prove (a). Note that we can assume that $q \notin \mathrm{Crit}(Y)$ for otherwise we replace q by some point in the stable manifold of q. Fix a small neighborhood U of $\mathrm{Cl}(W_Y^u(\sigma))$ such that $q \notin U$.

For each n denote $q_n = Y_{t_n}(x_n)$. As $x \in \mathrm{Cl}(W_Y^u(\sigma))$ there is $p \in W_Y^u(\sigma) \setminus \{\sigma\}$ satisfying the following property: For every $\delta > 0$ there is $t_p \geq 0$ and $x_p \in B_\delta(p)$ such that $X_{t_p}(x_p) \in B_\delta(x)$. Note that $p \notin \mathrm{Crit}(Y)$.

By (5) there is $t_q = -t_n$ and $x_q = Y_{t_n}(x_n)$ such that $X_{t_q}(x_q) \in B_\delta(x)$. Then, by Lemma 2.6, we find Z C^1 near to Y, $Z = Y$ outside a small compact neigbourhood of $Y_{[-L,L]}(x)$, for some large L, and such that $q \in \mathcal{O}_Z^+(p)$.

Since $p \in W_Z^u(\sigma(Z))$ and $q \notin U$, we obtain that $\mathrm{Cl}(W_Z^u(\sigma(Z)))$ is not contained in U, and thus Φ_i is not upper semi-continuous at Y, a contradiction. The proof of Lemma 4.2 is complete. □

Proof of Lemma 2.9. If $z \in \omega_X(z)$ (i.e. z is recurrent), then $\omega_X(z) = \mathrm{Cl}(\mathcal{O}_X^+(z))$ and we are done. So, we can assume that z is not recurrent. In particular, $z \notin Sing(X)$.

By contradiction, assume that $\mathrm{Cl}(\mathcal{O}_X^+(z))$ is Lyapunov stable and that $\omega_X(z)$ does not. Then there are a neighborhood U of $\omega_X(z)$ and a sequence $p_n \in M \to p \in \omega_X(z)$ such that $p_n' = X_{t_n}(p_n) \notin U$ for some $t_n \geq 0$.

Passing to a subsequence, if necessary, the limit $x = \lim_{n\to\infty} p_n'$ exists.

Since $\mathrm{Cl}(\mathcal{O}_X^+(z))$ is Lyapunov stable for X, we apply Lemma 2.1 to $\Lambda = \mathrm{Cl}(\mathcal{O}_X^+(z))$ and we obtain $x \in \mathrm{Cl}(\mathcal{O}_X^+(z)) \setminus U$.

Choose $T > 0$ depending on U such that $X_t(z) \in U$ for all $t \geq T$ (T exists since $\omega_X(z) \subset U$). Then, $x \in X_{[0,T]}(z) \setminus U$.

We consider a cross-section Σ containing z, $\delta > 0$ small and the flow box

$$B = X_{[-\delta, T+\delta]}(\Sigma).$$

Note that $W = U \cup B$ is a neighborhood of $\mathrm{Cl}(\mathcal{O}_X^+(z))$.

If δ and Σ are chosen small, then we have the following properties related to p_n, t_n and x as before: As $t_n \geq 0$ is a sequence such that $X_{t_n}(p_n) \to x$, then there is $t_n' \in [0, t_n]$ such that $p_n'' = X_{t_n'}(p_n) \notin B$. In another words, the positive trajectory of X through p_n must enter B before it passes close to x.

Passing to a subsequence if necessary, we can assume, as before, that $x' = \lim_{n\to\infty} p_n''$ exists. Note that $x' \notin W$. By Lemma 2.1 we obtain $x' \in \mathrm{Cl}(\mathcal{O}_X^+(z))$. But this is impossible since W is a neighborhood of $\mathrm{Cl}(\mathcal{O}_X^+(z))$. The proof is complete. □

References

[1] M. C. Arnaud, *Création de connexions en topologie* C^1. C. R. Acad. Sci. Paris, Série I, **329** (1999), 211–214.

[2] N.P. Bhatia and G.P. Szego, *Stability theory of dynamical systems*. Springer-Verlag, Heidelberg, 1979.

[3] C. Bonatti, A. Pumariño and M. Viana, *Lorenz attractors with arbitrary expanding dimension.* C. R. Acad. Sci. Paris, **325** (1997), Série I: 883–888.

[4] C. M. Carballo, C. A. Morales and M. J. Pacifico, *Homoclinic classes for generic* C^1 *vector fields.* Preprint MAT.07/2000 PUC-Rio, 2000.

[5] W. de Melo and J. Palis, *Geometric Theory of Dynamical Systems-An Introduction.* Springer Verlag, Berlin, 1982.

[6] S. Hayashi, *Connecting invariant manifolds and the solution of the* C^1 *stability and* Ω*-stability conjectures for flows.* Annals of Math., **145** (1997), 81–137.

[7] S. Hayashi, *Hyperbolicity, stability, and the creation of homoclinic points.* In Documenta Mathematica, Extra Volume ICM, Vol. II, 1998, 1998.

[8] M. Hirsch, C. Pugh and M. Shub, *Invariant manifolds*, volume 583 of Lect. Notes in Math. Springer Verlag, Berlin, 1977.

[9] M. Hurley, *Attractors: persistence, and density of their basins.* Trans. AMS, **269** (1982), 247–271.

[10] K. Kuratowski, *Topology II.* Academic Press- PWN-Polish Sci. Publishers Warszawa, 1968.

[11] C. Morales and M. J. Pacifico, Non-transitive attracting sets for 3-flows. In preparation.

[12] C. Morales and M. J. Pacifico, Lyapunov stability of generic ω-limit sets. Preprint, 2000.

[13] C. Morales, M. J. Pacifico and E. Pujals, *On* C^1 *robust singular transitive sets for three-dimensional flows.* C. R. Acad. Sci. Paris, **326** (1998), Série I: 81–86.

[14] C. Morales and E. Pujals, *Singular strange attractors on the boundary of Morse-Smale systems.* Ann. Sci. École Norm. Sup., **30** (1997), 693–717.

[15] D. V. Turaev and L. P. Shilnikov, *An example of a wild strange attractor.* Mat. Sbornik, **189** (1998), 137–160.

C. M. Carballo
Departamento de Matemática, PUC-Rio
Rio de Janeiro, RJ, Brazil

E-mail: carballo@mat.puc-rio.br

C. A. Morales, M. J. Pacifico
Instituto de Matemática
Universidade Federal do Rio de Janeiro
C. P. 68.530, CEP 21.945-970
Rio de Janeiro, RJ, Brazil

E-mail: morales@impa.br / pacifico@impa.br